

On the integral of the fourth Jacobi theta function

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Abstract

We generalize the Raabe-formula to the q -loggamma function. As a consequence, we get that the integral of the logarithm of the fourth Jacobi theta function between its least imaginary zeros is connected to the partition function and the Riemann zeta function.

Key words: q -gamma function, q -loggamma function, Jacobi theta functions, hypergeometric function, Riemann zeta function

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1 Introduction

The main result of the paper is the next integral formula for the fourth Jacobi theta function:

$$\int_{-x^*}^{x^*} \log \vartheta_4(x, q) dx = i \left[\zeta(2) - \log q \log \sum_{n=0}^{\infty} P(n) q^{2n} \right].$$

Here $x^* = \frac{1}{2}i \log q$ is the least zero (on the imaginary axis) for the Jacobi theta function

$$\vartheta_4(x, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nix},$$

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$P(n)$ is the partition function of the natural numbers [3], $0 < q < 1$ and $i = \sqrt{-1}$.

2 Preliminaries

2.1 Raabe's formula

In 1840 J. L. Raabe [12] proved that for the Euler Γ function

$$\int_0^1 \log \Gamma(x+t) dx = \log \sqrt{2\pi} + (t \log t - t) \quad (t \geq 0).$$

This implies the special case

$$\int_0^1 \log \Gamma(x) dx = \log \sqrt{2\pi},$$

and an immediate consequence is that

$$\int_0^1 \log \Gamma(x) \Gamma(1-x) dx = \log 2\pi.$$

(See [1] for an elementary proof of this special case.) We shall prove the appropriate integral formula for the Jackson q -gamma function. Then we show that the Jacobi triple product identity connects the q -gamma function to ϑ_4 and our main formula will follow.

2.2 The Jackson's q -gamma function

F. H. Jackson defined the q -analogue of the standard Euler $\Gamma(z)$ function as [6,11]

$$\Gamma_q(z) = \frac{(q^{-1}; q^{-1})_\infty}{(q^{-z}; q)_\infty} (q-1)^{1-z} \quad (q > 1) \quad (1)$$

with $(x; q)_\infty = (1-x)(1-qx)(1-q^2x) \cdots$.

2.3 The zeta regularized product

Let us consider a sequence $\mathbf{a} = (a_1, a_2, \dots)$. Its zeta regularized product is denoted and defined by [7,8]

$$\widehat{\prod_{n=1}^{\infty} a_n} = \exp(-\zeta'_{\mathbf{a}}(0)).$$

Here

$$\zeta'_{\mathbf{a}}(s) = \sum_{n=1}^{\infty} a_n^{-s}$$

is the zeta function associated with the sequence \mathbf{a} . It is assumed that $\zeta_{\mathbf{a}}(s)$ has an analytic continuation to a region containing $s = 0$ and further that it is holomorphic at this point.

M. Lerch [9] proved that for $\mathbf{a} = (1+x, 2+x, \dots)$

$$\widehat{\prod_{n=1}^{\infty} (n+x)} = \frac{\sqrt{2\pi}}{\Gamma(x)} \quad (x > 0).$$

This comes from the fact that the sequence \mathbf{a} above has the associated zeta function

$$\zeta_{\mathbf{a}}(s) = \zeta(s, x) = \sum_{n=1}^{\infty} (n+x)^{-s}$$

(which is the well known Hurwitz zeta function) and that

$$\zeta'(0, x) = \log \frac{\Gamma(x)}{\sqrt{2\pi}}.$$

Our considerations need a more general form of the zeta regularization. This was worked out in 2005 by N. Kurokawa and N. Wakayama [7]. They assumed that the $\zeta_{\mathbf{a}}(s)$ function is meromorphic at $s = 0$ with the Laurent expansion

$$\zeta_{\mathbf{a}}(s) = \sum_m c_m(\mathbf{a}) s^m.$$

Then the generalized zeta regularized product is defined by

$$\widehat{\prod_{n=1}^{\infty} a_n} = \exp(-c_1(\mathbf{a})) = \exp\left(-\operatorname{Res}_{s=0} \frac{\zeta_{\mathbf{a}}(s)}{s^2}\right). \quad (2)$$

See [10] for nice applications.

We introduce the short and standard notation

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

With this abbreviation the zeta function associated with the sequence $\mathbf{a} = ([x], [1+x]_q, [2+x]_q, \dots)$ is

$$\zeta_{\mathbf{a}}(s) = \zeta_q(s, x) = \sum_{n=0}^{\infty} [n+x]_q^{-s}.$$

Then the generalized zeta regularized product reads as

$$\widehat{\prod_{n=0}^{\infty} [n+x]_q} = \widehat{\prod_{n=1}^{\infty} \frac{q^{n+x} - 1}{q - 1}} = \frac{C_q}{\Gamma_q(x)} \quad (q > 1), \quad (3)$$

where

$$C_q = q^{-\frac{1}{12}} (q-1)^{\frac{1}{2} - \frac{\log(q-1)}{2 \log q}} (q; q)_{\infty}. \quad (4)$$

This is the second theorem of Kurokawa and Nakayama in [7] and this will be our main tool. (A more general form of this theorem is presented in [10].)

We split the presentation to two sections. The next one contains the generalized Raabe's formula, the other contains the proof of the integral formula of ϑ_4 .

3 Integral of the q -loggamma function

The q -analogue of Raabe's theorem is given:

Theorem 1 *For any $t > 0$ and $q > 1$*

$$\int_0^1 \log \Gamma_q(x+t) dx = \quad (5)$$

$$\begin{aligned} & \log C_q - \frac{1}{2q^t \log q} \left[\frac{1-q^t}{1-q^{-t}} (2 \operatorname{Li}_2(q^{-t}) + \log^2(1-q^{-t})) + \right. \\ & \left. 2 \frac{1-q^t}{1-q^{-t}} \log \frac{1-q}{1-q^t} \log(1-q^{-t}) - q^t \log^2 \frac{1-q}{1-q^t} \right]. \end{aligned}$$

In special,

$$\int_0^1 \log \Gamma_q(x) dx = \frac{\zeta(2)}{\log q} + \log \sqrt{\frac{q-1}{\sqrt[6]{q}}} + \log(q^{-1}; q^{-1})_{\infty}. \quad (6)$$

Here $\operatorname{Li}_2(z)$ is the dilogarithm function:

$$\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

To prove this theorem, we need the following Theorem, which is interesting itself.

Theorem 2 For all $q > 1$,

$$\int_0^1 \zeta_q(s, x+t) dx = \frac{(q-1)^s}{s \log q} \frac{(q^t-1)^{1-s}}{q^t} {}_2F_1(1, 1; s+1; q^{-t}).$$

Here

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is the hypergeometric function and $(a)_n = a(a+1) \cdots (a+n-1)$ is the Pochhammer symbol.

Proof of Theorem 2.

$$\begin{aligned} \int_0^1 \zeta_q(s, x+t) dx &= \int_0^1 \sum_{n=0}^{\infty} [n+x+t]_q^{-s} dx = \\ &= (q-1)^s \sum_{n=0}^{\infty} \int_0^1 (q^{n+x+t} - 1)^{-s} dx. \end{aligned}$$

This latter integral is computed by Wolfram Mathematica:

$$\begin{aligned} \int_0^1 (q^{n+x+t} - 1)^{-s} dx &= \frac{1}{s \log q} \left[\frac{(q^{n+t} - 1)^{1-s}}{q^{n+t}} {}_2F_1(1, 1; s+1; q^{-n-t}) - \right. \\ &\quad \left. \frac{(q^{n+t+1} - 1)^{1-s}}{q^{n+t+1}} {}_2F_1(1, 1; s+1; q^{-n-t-1}) \right]. \end{aligned}$$

Since

$$\begin{aligned} {}_2F_1(1, 1; s+1; q^{-n-t}) &= \sum_{k=0}^{\infty} \frac{k!}{(s+1)_k} \frac{1}{(q^{n+t})^k}, \\ \sum_{n=0}^{\infty} \int_0^1 (q^{n+x+t} - 1)^{-s} dx &= \\ \frac{1}{s \log q} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{k!}{(s+1)_k} &\left(\frac{(q^{n+t} - 1)^{1-s}}{(q^{n+t})^{k+1}} - \frac{(q^{n+t+1} - 1)^{1-s}}{(q^{n+t+1})^{k+1}} \right). \end{aligned}$$

If we interchange the order of the summation, we see that the sum over n is telescopic, so the only one term which is not cancels belongs to $n = 0$. Thus the above expression simplifies to

$$\frac{1}{s \log q} \sum_{k=0}^{\infty} \frac{k!}{(s+1)_k} \frac{(q^t - 1)^{1-s}}{(q^t)^{k+1}} = \frac{(q^t - 1)^{1-s}}{s q^t \log q} \sum_{k=0}^{\infty} \frac{k!}{(s+1)_k} \frac{1}{(q^t)^k}.$$

This latter sum is again hypergeometric with parameters $(1, 1; s+1; q^{-t})$ and we get our Theorem.

Proof of Theorem 1. (2) and (3) together gives that

$$\int_0^1 \log \Gamma_q(x+t) dx = \log C_q + \int_0^1 \operatorname{Res}_{s=0} \frac{\zeta(s, x+t)}{s^2} dx.$$

Since the residue is taken with respect to s , we can carry out before the integral. Hence, by Theorem 2,

$$\int_0^1 \log \Gamma_q(x+t) dx = \log C_q + \operatorname{Res}_{s=0} \frac{(q-1)^s (q^t-1)^{1-s}}{s^3 \log q} \frac{1}{q^t} {}_2F_1(1, 1; s+1; q^{-t}).$$

The residue can be calculated with Mathematica. It equals to

$$\begin{aligned} & \frac{-1}{2q^t \log q} \left[(1-q^t) \frac{\partial^2}{\partial s^2} {}_2F_1(1, 1; s; q^{-t})_{s=1} + \right. \\ & \left. 2(1-q^t) \log \frac{1-q}{1-q^t} \frac{\partial}{\partial s} {}_2F_1(1, 1; s; q^{-t})_{s=1} - q^t \log^2 \frac{1-q}{1-q^t} \right]. \end{aligned} \quad (7)$$

Now we deal with the partial derivatives. Simbolically,

$$\frac{\partial^n}{\partial s^n} {}_2F_1(1, 1; s; z) = \sum_{n=0}^{\infty} (a)_n (b)_n \frac{\partial^n}{\partial s^n} \frac{1}{(c)_n} \frac{z^n}{n!}. \quad (8)$$

The Pochhammer symbol can be rewritten with the Γ function:

$$(c)_n = \frac{\Gamma(c+n)}{\Gamma(c)},$$

whence

$$\frac{\partial}{\partial s} \frac{1}{(c)_n} = \frac{-1}{(c)_n} (\psi(c+n) - \psi(c)), \quad (9)$$

and

$$\frac{\partial^2}{\partial s^2} \frac{1}{(c)_n} = \frac{(\psi(c+n) - \psi(c))^2}{(c)_n} - \frac{\psi'(c+n) - \psi'(c)}{(c)_n}.$$

Here

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is the digamma function. When n is a positive integer, then [2, p. 13]

$$\psi(n) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} = H_{n-1} - \gamma, \quad (10)$$

and

$$\psi'(n) = -\frac{1}{1^2} - \frac{1}{2^2} - \cdots - \frac{1}{(n-1)^2} + \zeta(2) = -H_{n-1,2} + \zeta(2).$$

(H_n and $H_{n,2}$ are the harmonic and second order harmonic numbers, respectively. $H_0 = H_{0,2} = 0$.) Now (8), (9) and (10) gives that

$$\frac{\partial}{\partial s} {}_2F_1(1, 1; s; z)_{s=1} = \sum_{n=0}^{\infty} n!n! \frac{-H_n z^n}{n! n!} = - \sum_{n=0}^{\infty} H_n z^n = \frac{\log(1-z)}{1-z}.$$

Similarly, for the second order derivative

$$\frac{\partial^2}{\partial s^2} {}_2F_1(1, 1; s; z)_{s=1} = \sum_{n=0}^{\infty} n!n! \left(\frac{H_n^2}{n!} + \frac{H_{n,2}}{n!} \right) \frac{z^n}{n!} = \sum_{n=1}^{\infty} H_n^2 z^n + \sum_{n=1}^{\infty} H_{n,2} z^n. \quad (11)$$

By Cauchy's product, the latter sum is simply

$$\frac{1}{1-z} \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \frac{\text{Li}_2(z)}{1-z}.$$

The first sum can be determined easily. Note that

$$H_{n-1}^2 = \left(H_n - \frac{1}{n} \right)^2 = H_n^2 + \frac{1}{n^2} - 2 \frac{H_n}{n},$$

whence

$$\sum_{n=1}^{\infty} H_{n-1}^2 z^n = \sum_{n=1}^{\infty} H_n^2 z^n + \sum_{n=1}^{\infty} \frac{z^n}{n^2} - 2 \sum_{n=1}^{\infty} \frac{H_n}{n} z^n. \quad (12)$$

The last sum equals to [4]

$$\sum_{n=1}^{\infty} \frac{H_n}{n} z^n = \text{Li}_2(z) + \frac{1}{2} \log^2(1-z). \quad (13)$$

If we temporarily introduce the function

$$f(z) = \sum_{n=1}^{\infty} H_n^2 z^n,$$

then (12) and (13) implies that

$$zf(z) = f(z) + \text{Li}_2(z) - 2(\text{Li}_2(z) + \frac{1}{2} \log^2(1-z)),$$

hence

$$f(z) = \sum_{n=1}^{\infty} H_n^2 z^n = \frac{\text{Li}_2(z) + \log^2(1-z)}{1-z}.$$

Altogether, (11) becomes

$$\frac{\partial^2}{\partial s^2} {}_2F_1(1, 1; s; z)_{s=1} = \frac{2 \text{Li}_2(z) + \log^2(1-z)}{1-z}.$$

The partial derivatives in (7) are determined and the first part of the Theorem is proved.

If $t \rightarrow 0$, our expression under (5) simplifies:

$$\begin{aligned}
& \lim_{t \rightarrow 0} \frac{-1}{2q^t \log q} \left[\frac{1 - q^t}{1 - q^{-t}} (2 \operatorname{Li}_2(q^{-t}) + \log^2(1 - q^{-t})) + \right. \\
& \quad \left. 2 \frac{1 - q^t}{1 - q^{-t}} \log \frac{1 - q}{1 - q^t} \log(1 - q^{-t}) - q^t \log^2 \frac{1 - q}{1 - q^t} \right] = \\
& \frac{-1}{2 \log q} \left[-2 \operatorname{Li}_2(1) - \lim_{t \rightarrow 0} \left(+ \log^2(1 - q^{-t}) + 2 \log \frac{1 - q}{1 - q^t} \log(1 - q^{-t}) + \log^2 \frac{1 - q}{1 - q^t} \right) \right] = \\
& \frac{-1}{2 \log q} \left[-2 \operatorname{Li}_2(1) - \lim_{t \rightarrow 0} \left(\log(1 - q^{-t}) + \log \frac{1 - q}{1 - q^t} \right)^2 \right] = \\
& \frac{-1}{2 \log q} \left[-2 \operatorname{Li}_2(1) - \lim_{t \rightarrow 0} \log^2 \left((1 - q) \frac{1 - q^{-t}}{1 - q^t} \right) \right] = \frac{1}{2 \log q} (2\zeta(2) + \log^2(q - 1)).
\end{aligned}$$

Thus (5) tends to the simple expression

$$\int_0^1 \log \Gamma_q(x) dx = \log C_q + \frac{1}{2 \log q} (2\zeta(2) + \log^2(q - 1)).$$

The definition (4) of C_q enables us to get a more simple identity. Since

$$\log C_q = -\frac{1}{12} \log q + \frac{1}{2} \log(q - 1) - \frac{\log^2(q - 1)}{2 \log q} + \log(q^{-1}; q^{-1})_\infty,$$

the term $\frac{\log^2(q-1)}{2 \log q}$ cancels and a trivial modification gives the second formula (6) of our Theorem.

4 The proof of the main formula

The (1) definition of the q -Gamma function and some reduction gives that for any $q > 1$ and $y > 0$

$$\begin{aligned}
& \frac{1}{\Gamma_{q^2} \left(\frac{1}{2} \log_q \frac{q}{y} \right) \Gamma_{q^2} \left(\frac{1}{2} \log_q qy \right)} = \\
& \frac{(q^{\frac{1}{2}})^{1 - \log_q^2 y}}{(q^{-2}; q^{-2})_\infty^3 (q^2 - 1)} (q^{-2}; q^{-2})_\infty (y/q; q^{-2})_\infty (1/(yq); q^{-2})_\infty
\end{aligned} \tag{14}$$

This product can be rewritten by Jacobi's triple product identity [5, p. 15]:

$$(q^2; q^2)_\infty (qy; q^2)_\infty (q/y; q^2)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} y^n,$$

or, which is the same,

$$(q^{-2}; q^{-2})_{\infty} (y/q; q^{-2})_{\infty} (1/(qy); q^{-2})_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{-n^2} y^n.$$

In (14) we choose $y = q^{1-2x}$. Then $\frac{1}{2} \log_q \frac{q}{y} = x$ and $\frac{1}{2} \log_q qy = 1 - x$, so Jacobi's identity yields

$$\frac{1}{\Gamma_{q^2}(x) \Gamma_{q^2}(1-x)} = \frac{(q^{\frac{1}{2}})^{1-(1-2x)^2}}{(q^{-2}; q^{-2})_{\infty}^3 (q^2 - 1)} \sum_{n=-\infty}^{\infty} (-1)^n q^{-n^2} (q^{1-2x})^n.$$

Consider the definition of the ϑ_4 function on the first page. It is not hard to see that we arrive at the next formula:

$$\frac{1}{\Gamma_{q^2}(x) \Gamma_{q^2}(1-x)} = \frac{q^{2x(1-x)}}{(q^{-2}; q^{-2})_{\infty}^3 (q^2 - 1)} \vartheta_4 \left(\frac{1}{2i} (1-2x) \log q, \frac{1}{q} \right).$$

In the next step we take logarithm of both sides and integrate on $[0, 1]$.

$$\int_0^1 \log \Gamma_{q^2}(x) \Gamma_{q^2}(1-x) dx =$$

$$\log(q^{-2}; q^{-2})_{\infty}^3 (q^2 - 1) - \log q \int_0^1 2x(1-x) dx - \int_0^1 \log \vartheta_4 \left(\frac{1}{2i} (1-2x) \log q, \frac{1}{q} \right) dx.$$

Using Theorem 1, the right hand side equals to

$$\frac{2\zeta(2)}{\log q^2} + \log \frac{q^2 - 1}{\sqrt[6]{q^2}} + \log(q^{-2}; q^{-2})_{\infty}^2.$$

An elementary simplification implies that

$$\int_0^1 \log \vartheta_4 \left(\frac{1}{2i} (1-2x) \log q, \frac{1}{q} \right) dx = \log(q^{-2}; q^{-2})_{\infty} - \frac{\zeta(2)}{\log q}.$$

We transform the integral:

$$\int_0^1 \log \vartheta_4 \left(\frac{1}{2i} (1-2x) \log q, \frac{1}{q} \right) dx = \frac{-i}{\log q} \int_{-\frac{1}{2}i \log q}^{\frac{1}{2}i \log q} \log \vartheta_4 \left(x, \frac{1}{q} \right) dx.$$

Therefore

$$\int_{-\frac{1}{2}i \log q}^{\frac{1}{2}i \log q} \log \vartheta_4 \left(x, \frac{1}{q} \right) dx = \frac{1}{i} [\zeta(2) - \log q \log(q^{-2}; q^{-2})_{\infty}]. \quad (15)$$

Let us consider the endpoint of the integration:

$$\vartheta_4 \left(-\frac{1}{2}i \log q, \frac{1}{q} \right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n-n^2}.$$

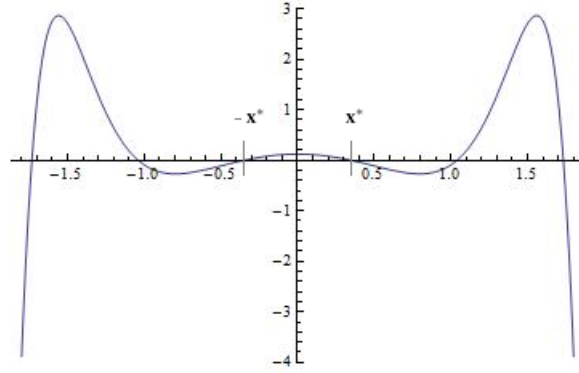
It is straightforward to see that all terms cancels, so

$$\vartheta_4 \left(-\frac{1}{2}i \log q, \frac{1}{q} \right) = 0.$$

Similarly,

$$\vartheta_4 \left(\frac{1}{2}i \log q, \frac{1}{q} \right) = 0.$$

Let $x^* = \frac{1}{2}i \log q$. To visualize the roots, we draw $\vartheta_4(ix, 1/2)$:



Note that until this point $q > 1$. Our formula will be more elegant if we substitute $q = \frac{1}{q}$. But in this case (15) modifies:

$$\int_{x^*}^{-x^*} \log \vartheta_4(x, q) dx = \frac{1}{i} \left[\zeta(2) + \log q \log(q^2; q^2)_\infty \right].$$

Interchanging the limits of the integration and using the well known generating function [3]

$$\sum_{n=0}^{\infty} P(n)q^n = (q; q)_\infty^{-1},$$

we are done.

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